Transport in a random medium with spatial correlations

B. Gaveau,[1] M. A. Gaveau,[2] A. Mogutov,[3] and M. Moreau[3]
[1]Mathematiques Pures et Appliquees, case 172, Université Pierre et Marie Curie, 75252 Paris Cedex 05, France
[2]Commissariat à Energie Atomique Saclay, 91191 Gif-sur-Yvette, France
[3]Laboratoire de Physique Théorique des Liquides, case 121, Université Pierre et Marie Curie, 75252 Paris Cedex 05, France
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We consider the transport properties of a particle moving on a one-dimensional lattice with nearest neighbor correlations between the states of the sites. In particular, we compute analytically the transmission probability and the autocorrelation of the velocity, and we prove that they have an anomalously slow decay.

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I. INTRODUCTION

The diffusion of a particle in a static random medium has been the subject of many works. Usually, the particle moves between different random scatterers with which it interacts according to given rules. The states of the scatterers are random variables which are, in general, independent from each other and completely static. The typical models which have been investigated are the diffusion of a particle in a Lorentz gas (the particle moves freely between scatterers until it hits one of them and change velocity) or the diffusion of a particle in a static random potential with or without velocities (see [1] for a nice review and [2–4], among many other references). Several one-dimensional models of diffusion in a frozen disorder have been studied, in which the velocity autocorrelation function presents a long time behavior in $t^{-\alpha}$ determined by irregular tails in the static distribution of scattering properties (for example, the residence times or the lengths of free intervals between scatterers). The exponent α then results directly from the law chosen for this distribution (see [5]). In the case of a Lorentz gas, the velocity autocorrelation function of the diffusing particles also has a tail in $t^{-\alpha}$ with $\alpha=d/2+1$, but this behavior is due to dynamically correlated collision rings and not to correlations present in the medium itself.

On the other hand, situations where the different states of the scatterers are correlated or fluctuate in time have not been very much investigated. Recently, we introduced models where the sites are fluctuating independently in time and we proved the existence of a diffusion constant, although the motion is not Gaussian (because of an abnormal kurtosis) (see [6–8] and [9]).

In this paper, we consider the case where the states of the sites are spatially correlated, by short range forces. This situation is obviously important for the diffusion of a particle in a solid or a liquid [10], when the time scale of the diffusing particle is fast compared to the relaxation time of the solid medium. In both cases, it is rather unrealistic to assume that the states of neighboring scatterers are independent random variables. On the contrary, they are rather strongly correlated random variables. As we shall see, although the forces in the solid are short range, a coupling between the sites due to these forces induces immediately an abnormal diffusion, with a long tail decay of the velocity, and a higher probability of transmission than in the usual diffusion processes. In the present model, the long time decay of the velocity autocorrelation function and the abnormal transmission rate comes form the Boltzmann distribution of the scatterers. The effect mentioned above is the result of the short range correlation between the scatterers due to the physical picture of the interactions.

In fact, our motivation comes from a model of diffusion in a solid. We describe in Sec. II the physical problem, and simplify it in Sec. III in order to obtain a solvable model. Sections IV and V give the behavior of the transmission probabilities and the decay of the velocity. Our results are analytically exact and involve no approximation beyond the simplification done in Sec. III.

II. DESCRIPTION OF THE MODEL

The general idea of the model is that of a diffusion of a particle P in a harmonic solid. We consider a linear lattice of sites labeled n (integers). At each site of the lattice, a scatterer can move transversally to the lattice. The state of a scatterer at n is a certain number q_n which measures the distance of the scatter to the lattice (see Fig. 1). The scatterers interact via near-neighbor harmonic interaction. The corresponding potential is

$$V(\{q_n\})=\frac{K}{2}\sum_n (q_n-q_{n-1})^2 . \tag{1}$$

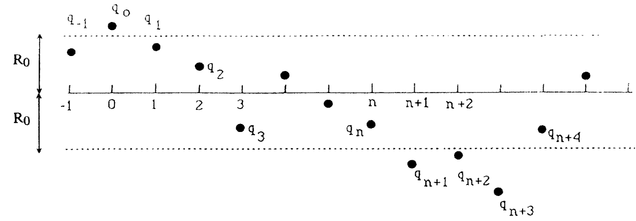

FIG. 1. Initial model: Solid circles, correlated scatterers at positions (n,q_n). A particle propagates on the straight line $0x$ and interact with scatterers if their distance with $0x$ is $\leq R_0$.

We assume that a configuration $\{q_n\}$ of the scatterers is fixed and chosen according to the Boltzmann distribution $\propto \exp(-\beta V)$.

Now we consider a particle P moving on the linear lattice with a velocity $v=\pm1$, according to the following rules: When P arrives at a site n with a certain velocity v, if the scatterer q_n is far from the lattice ($|q_n|>R_0$, for some given R_0), the scatterer does not interact with the particle; P keeps its velocity unchanged and goes to the neighboring site in the next time step. On the other hand, when q_n is near the lattice ($|q_n|\le R_0$), the scatterer interacts with P and, as a consequence, it changes the velocity of P with probability p and leaves it unchanged with probability $1-p$.

We will prove that, although the scatterers are only spatially correlated by near-neighbor interaction through the potential V, the motion of the particle has anomalous properties with respect to the usual diffusive behavior. More precisely, we show in Sec. IV that the transmission probability to get out through L from an interval $[0,L]$ starting from 0 with positive velocity varies as $L^{-1/2}\ln L$ for large L (instead of L^{-1} in the usual diffusion). In Sec. V we find that the expectation of the velocity decays like $t^{-1/2}$ for large t (instead of the exponential decay).

III. SIMPLIFIED DESCRIPTION OF THE ENVIRONMENT

In order to treat the model analytically, we need a further simplification. The successive increments $q_{n+1}-q_n$ of the states of the scatterers are independent Gaussian variables because of the distribution $\exp(-\beta V)$. Consequently, q_n can be viewed as the position at "time n" of a random walker with successive increments independently and Gaussianly distributed. We now make a further approximation and assume that q_n is a position at "time n" of a standard random walk on a lattice with spatial distance between two neighboring sites by $(K\beta)^{-1/2}$ (because, essentially, the increment $q_{n+1}-q_n$ is a Gaussian variable with that variance).

The important feature of the situation is the following: The lattice will be partitioned in a succession of stochastic intervals, numbered with positive or negative integers (see Fig. 2)

$$\dots, I_{-2}, I_{-1}, I_0, I_1, I_2, \dots \qquad (2)$$

such that in an odd-numbered interval such as

$$I_1, I_2, I_3, \dots, I_{2k+1}, \dots,$$

the scatterers q_n are far from the lattice ($|q_n|>R_0$) and the particle moves freely and in an even-numbered interval such as $I_{-2}, I_0, \dots, I_{2k}, \dots$, the scatterers q_n are near the lattice ($|q_n|\le R_0$) and they interact with the particle.

Let us denote by L_k the "length" of I_k, or, more exactly, the number of scatterers. The L_k are independent random variables which approximately obey the following continuous distributions.

(i) For the odd-numbered intervals, the distribution probability of the length is

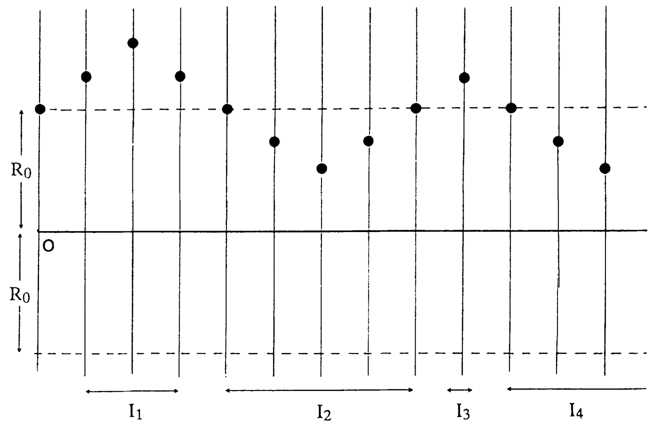

FIG. 2. Simplified model: q_n performs a random walk. Odd-numbered intervals contain scatterers q_n such that $q_n>R_0$; particles do not interact with them (ballistic motion). Even-numbered intervals contain scatterers q_n such that $q_n\le R_0$; a particle interacts with them and diffuses (random telegraph motion).

$$\mathrm{Prob}(L_{2k+1}\in[l,l+dl])=\lambda(l)dl , \qquad (3)$$

and $\lambda(l)$ has a long tail in $l^{-3/2}$ for large l (because it is the law of the return time of a random walk to a given point R_0 starting from a point $>R_0$) (see [11]). In particular, $\lambda(l)dl$ has *no finite moment.*

(ii) For the even-numbered intervals, the distribution probability of the length is

$$\mathrm{Prob}(L_{2k}\in[l,l+dl])=\mu(l)dl , \qquad (4)$$

and here $\mu(l)$ has moments of all orders (because it is the law of the exit time of a random walk from an interval $[-R_0,+R_0]$ starting inside this interval).

With these notations, we describe our simplified model. We have a line divided into a succession of intervals as in (2). The distribution laws of the lengths of thee intervals are given by Eq. (3) or Eq. (4). When the particle P is in an odd-numbered interval, it has a ballistic motion with velocity $v=\pm1$. When it is in an even-numbered interval, it has a random telegraph motion (namely, the velocity v changes sign at exponential times distributed with an exponential law $e^{-\chi t}$. To complete the description, we assume that, at time $t=0$, the particles starts with velocity $+1$ at point 0 and that 0 is the left extremity of I_1 (see Fig. 2).

IV. TRANSMISSION PROBABILITY

In this section we consider the probability $P(L)$ that the particle starting at 0 (thus at the beginning of I_1) with velocity $+1$ gets out for the first time from the interval $[0,L]$ through L rather than 0. We first notice that the transmission probability for a random telegraph process of frequency χ in an interval of length L is

$$p(L)=\frac{1}{1+\chi L} . \qquad (5)$$

We now distinguish three cases.

(i) L is in I_1. Then the particle gets out through L with probability 1. In computing $P(L)$, this event gives a contribution

$$\mathrm{Prob}(L \le L_1) \,. \tag{6}$$

(ii) L is in I_{2k+1} (for $k \ge 1$). The transmission probability knowing the lengths of the intervals is exactly $p(L_2+\cdots+L_{2k})$. This event gives a contribution

$$P_{2k}(L)=p(L_2+\cdots+L_{2k})\times\mathrm{Prob}\{L_1+\cdots+L_{2k}\le L\le L_1+\cdots+L_{2k+1}\} \,. \tag{7}$$

(iii) L is in L_{2k} (for $k\le 1$). Let us denote $q(L_1,\ldots,L_{2k},L)$ the transmission probability knowing the lengths of the intervals. We have obviously

$$p(L_2+\cdots+L_{2k})\le q(L_1,\ldots,L_{2k},L)\le p(L_2+\cdots+L_{2k-2}) \,. \tag{8}$$

This event gives a contribution

$$Q_{2k-1}(L)=q(L_1,\ldots,L_{2k},L)\times\mathrm{Prob}\{L_1+\cdots+L_{2k-1}\le L\le L_1+\cdots+L_{2k}\} \,, \tag{9}$$

so that

$$P(L)=\mathrm{Prob}(L\le L_1)+\sum_{k\ge 1}Q_{2k-1}(L)+\sum_{k\ge 1}P_{2k}(L) \,. \tag{10}$$

Moreover, it is easily found that

$$P_{2k}(L)=\int_0^L\lambda(l_1)dl_1\int_0^{L-l_1}\mu(l_2)dl_2\int_0^{L-l_1-l_2}\lambda(l_3)dl_3\cdots\int_0^{L-l_1-l\cdots-l_{2k-1}}\mu(l_{2k})dl_{2k}\times\int_{L-l_1-\cdots-l_{2k}}^{\infty}\lambda(l_{2k+1})dl_{2k+1}\times p(l_1,\ldots,l_{2k}) \,, \tag{11}$$

$$Q_{2k-1}(L)=\int_0^L\lambda(l_1)dl_1\int_0^{l-l_1}\mu(l_2)dl_2\int_0^{L-l_1-l_2}\lambda(l_3)dl_3\cdots\int_0^{L-l_1-l\cdots-l_{2k-2}}\lambda(l_{2k-1})dl_{2k-1}\times\int_{L-l_1-\cdots-l_{2k}}^{\infty}\mu(l_{2k})dl_{2k}\times q(l_1,\ldots,l_{2k},L) \,. \tag{12}$$

In Appendix A we prove that for L large, we have

$$P(L)\propto\left|\frac{\ln L}{L^{1/2}}\right| \,, \tag{13}$$

which shows that the transmission probability decays much more slowly with L than in the usual diffusion.

V. DECAY OF THE VELOCITY CORRELATION FUNCTION

A. Velocity autocorrelation function in the general case

Let us assume that we start at point $x=0$, with velocity $v(0)=+1$. We also assume that the site $x=0$ is the left extremity of the interval I_1. Given these conditions, the velocity autocorrelation function (see [10]) is just $\langle v(t)\rangle$. In Fig. 3 we present a numerical simulation of the slow decay of $\langle v(t)\rangle$, namely, for large t:

$$\langle v(t)\rangle\approx\frac{C}{t^{1/2}} \,, \tag{14}$$

where C is some positive constant. This simulation was done with the initial model described in Sec. II, rather than the simplified model, with 10^3 barriers until about 2×10^3 time steps.

We have not been able to prove this result analytically. Nevertheless, we now present some analytic work which rigorously confirms the slow decay.

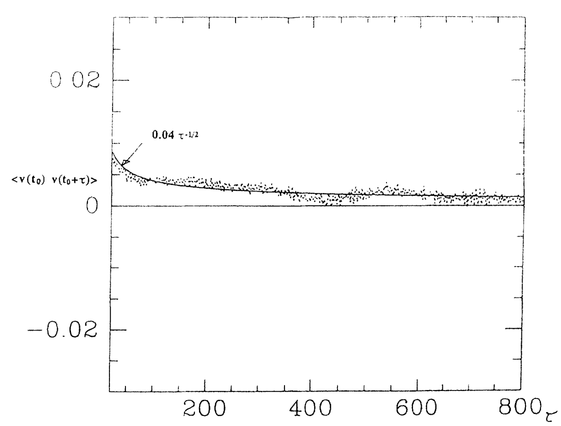

FIG. 3. Velocity correlation function for random telegraph process (RTP) in the intervals of interaction: Dots, numerical results; solid line, best fit with curves $K\tau^{-1/2}$ ($K\sim0.04$). RTP relaxation frequency, 0.2; $R_0=10$; lattice spacing, 1; absolute velocity, 1.

B. Strong field case

Let us come back to the simplified model of Sec. III and let us also impose a strong field pointing to the right. The action of this field on the particle is the following: When the particle is near a scatterer (in the even-numbered intervals), it is only subjected to the action of the scatterer and does not feel the field. On the other hand, when the particle is far from the scatterer (odd-numbered intervals), it always has a ballistic motion with uniform velocity $v=+1$. In particular, when the particle is in an even-numbered interval I_{2k} where it follows a random telegraph process, it can only exit through the right extremity of the interval, because as soon as it enters the odd-numbered interval I_{2k-1} coming from the right, it feels the field and is pushed back in I_{2k}. It is clear that $\langle v(t)\rangle$ tends to 1 when t tends to infinity. In an ordinary random telegraph process, $\langle v(t)\rangle-1$ would decay exponentially fast like $e^{-\chi t}$ where χ is the frequency of the random telegraph process. Here we shall prove that for large t,

$$\langle v(t)\rangle - 1 \approx \frac{K}{\sqrt{t}} \tag{15}$$

for some negative constant K that we can actually compute explicitly. This confirms the numerical simulations in the absence of the strong field.

We briefly sketch the mathematical arguments, leaving the details to Appendix B. First, when the particle is in an odd-numbered interval I_{2k+1}, the exit time starting from the left extremity of I_{2k+1} is just L_{2k+1}. Moreover, when the particle starts from the left extremity of I_{2k}, the exit time T_{2k} from I_{2k} is the exit time of a random telegraph process from an interval I_{2k} of length L_{2k} with a reflecting barrier at the left extremity of I_{2k} (because the strong field prevents the particle from reentering an already visited odd-numbered interval). Let us denote

$$V(t)=\langle v(t)\eta(\{t\le T_{2k}\})\rangle , \tag{16}$$

$\eta(E)$ being the characteristic function of an event E: $\eta(E)=1$ if E is realized, and 0 otherwise. Thus, $V(t)$ is the expectation value of the velocity at time t during the motion in I_{2k}, if the particle starts from the left extremity of I_{2k}. Let us also denote

$$\tau(t)dt=\text{Prob}(T_{2k}\in[t,t+dt]) . \tag{17}$$

As for (10), we have

$$\langle v(t)\rangle=\text{Prob}(T\le L_1)+\sum_{n=1}^{\infty}\text{Prob}(L_1+T_2+L_3+\cdots+T_{2n}\le t\le L_1+T_2+\cdots+T_{2n}+L_{2n+1})$$
$$+\sum_{n\ge 1}\langle v(t)\eta\{L_1+T_2+\cdots+L_{2n-1}\le t\le L_1+\cdots+L_{2n-1}+T_{2n}\}\rangle . \tag{18}$$

It is easy to see that

$$\text{Prob}(L_1+T_2+\cdots+T_{2n}\le t\le L_1+T_2+\cdots+T_{2n}+L_{2n+1})=(\Lambda\circ\tau\circ\lambda\circ\tau\circ\cdots\circ\lambda)(t) , \tag{19}$$

where we have n times τ and n times λ, $\circ$ denoting convolution, and where we have written

$$\Lambda(l)=\int_l^\infty \lambda(l)dl . \tag{20}$$

Moreover,

$$\langle v(t)\eta\{L_1+T_2+\cdots+L_{2n-1}\le t\le L_1+\cdots+L_{2n-1}+T_{2n}\}\rangle=(V\circ\tau\circ\lambda\circ\tau\circ\cdots\circ\lambda)(t) , \tag{21}$$

with n times λ and $n-1$ times τ. From (18), (19), and (21), we see that the Laplace transform of $\langle v(t)\rangle$ is

$$\int_0^\infty e^{-\sigma t}\langle v(t)\rangle dt=\frac{\hat{\Lambda}(\sigma)+\hat{\lambda}(\sigma)\hat{V}(\sigma)}{1-\lambda(\sigma)\tau(\sigma)} . \tag{22}$$

In Appendix B it is proved that for large t,

$$\langle v(t)\rangle-1\approx\frac{\langle L_{2k}\rangle-\langle T_{2k}\rangle}{K\sqrt{\pi}}\frac{1}{\sqrt{t}} , \tag{23}$$

where

$$\hat{\lambda}(\sigma)=e^{-K\sqrt{\sigma}} \quad \text{(see [10], for example)} ,$$
$$\langle L_{2k}\rangle=\int_1^\infty l\mu(l)dl ,$$
$$\langle T_{2k}\rangle=\int_1^\infty \mu(l)\langle T_{2k}|L_{2k}=l\rangle dl ,$$

and $\langle T_{2k}|L_{2k}=l\rangle$ is the conditional expectation of the exit time from $[0,L_{2k}]$ of the random telegraph particle starting from 0 at time 0 and reflecting back at 0, knowing that $L_{2k}=l$. We notice that in Eq. (23), $\langle L_{2k}\rangle-\langle T_{2k}\rangle$ is indeed negative, because the expected exit time of the interval I_{2k} of length L_{2k}, starting from its extremity and with reflexion at this extremity, is greater than the expected length.

In Appendix C we evaluate the constant in the second member of (23) and we prove that for large R_0,

$$\langle v(t)\rangle-1\sim-\frac{R_0^3}{\sqrt{t}} .$$

This is obviously not uniform in t. In fact, if R_0 was infinite, the particle would experience a standard random walk in the lattice of integers, because the diffusing particle would interact with every scatterer, and the diffusion would be normal.

This result, Eq. (23) or Eq. (14), expresses a slow decay law as in one-dimensional hydrodynamics (see [10]), but here the physical origin of this decay law is not the same as in hydrodynamics. In hydrodynamics, the slow decay is essentially due to the backflow built around the moving particle according to the Navier-Stokes equation. In our context, we have no such concepts and the decay is essentially controlled by the fluctuations of the environment. It should also be noted that similar effects often appear in the properties of transport or reactivity of disordered media. In particular, the spatial fluctuations of the medium have been shown to slow down the kinetics of various model chemical reactions [12]. It should also be noted that our model can be considered as a model with effectively finite size scatterers (the I_{2k} intervals) separated by free intervals of length L_{2k+1} with a length distribution $\lambda(l)\sim\lambda^{-\beta}$ for large l. The calculations of Appendix B then yield a velocity autocorrelation function in $t^{\beta-2}$. In the present model, the value $\beta=\frac{3}{2}$ comes out naturally from the Boltzmann distribution of the scatterers.

ACKNOWLEDGMENT

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APPENDIX A: ESTIMATION OF THE TRANSMISSION PROBABILITY

Let us define, with the notations of Sec. II,

$$\hat{P}_{2k}(\sigma)=\int_0^\infty e^{-\sigma L}P_{2k}(L)dL \ .$$

We notice that from (5),

$$p(L)=\int_0^\infty e^{-\theta(1+\chi L)}d\theta \ ,$$

so that from (11), the Laplace transform $\hat{P}_{2k}(\sigma)$ is

$$\hat{P}_{2k}(\sigma)=\int_0^\infty e^{-\theta}[\hat{\lambda}(\sigma)\hat{\mu}(\sigma+\chi\theta)]^k\hat{\Lambda}(\sigma)d\theta \ ,$$

where $\Lambda(l)$ is defined as in (20).

In the same way, using (9) and (8), we obtain

$$\int_0^\infty e^{-\theta}[\hat{\lambda}(\sigma)\hat{\mu}(\sigma+\chi\theta]^k\hat{M}(\sigma+\chi\theta)d\theta \leq \hat{Q}_{2k-1}(\sigma)\leq \int_0^\infty e^{-\theta}[\hat{\lambda}(\sigma)\hat{\mu}(\sigma+\chi\theta)]^{k-1}\times\hat{M}(\sigma+\chi\theta)d\theta \ ,$$

where $M(l)=\int_l^\infty \mu(l)dl$. From (10) we obtain

$$\hat{P}(\sigma)=\hat{\Lambda}(\sigma)\int_0^\infty \frac{e^{-\theta}}{1-\hat{\lambda}(\sigma)\mu(\sigma+\chi\theta)}d\theta+\sum_{k\geq 1}\hat{Q}_{2k-1}(\sigma) \ , \tag{A1}$$

with

$$\hat{\Lambda}(\sigma)\int_0^\infty e^{-\theta}\frac{\hat{\mu}(\sigma+\chi\theta)\hat{M}(\sigma+\chi\theta)}{1-\lambda(\sigma)\mu(\sigma+\chi\theta)}d\theta \leq \hat{Q}_{2k-1}(\sigma)\leq \hat{\Lambda}(\sigma)\int_0^\infty e^{-\theta}\frac{\hat{M}(\sigma+\chi\theta)}{1-\lambda(\sigma)\mu(\sigma+\chi\theta)} \ . \tag{A2}$$

We must now estimate each term in (A1) for small σ.

1. Estimation of the first sum in (A1)

We have

$$\hat{\Lambda}(\sigma)=\frac{1}{\sigma}[1-\hat{\lambda}(\sigma)]\approx\frac{K}{\sqrt{\sigma}} \ ,$$

because $\hat{\lambda}(\sigma)\approx e^{-K\sqrt{\sigma}}$ for small σ (see [10]). Then,

$$\hat{\Lambda}(\sigma)\int_0^\infty e^{-\theta}\frac{1}{1-\hat{\lambda}(\sigma)\mu(\sigma+\chi\theta)} \approx\frac{K}{\sqrt{\sigma}}\int_0^\infty e^{-\theta}\frac{d\theta}{1-\hat{\mu}(\chi\theta)+K\sqrt{\sigma}\hat{\mu}(\chi\theta)} \ .$$

This integral can be split into two parts:

$$\int_0^\varepsilon+\int_\varepsilon^{+\infty} \ .$$

For $\sigma=0$, the integral $\int_\varepsilon^{+\infty}$ is finite.

Moreover, for small σ, we have

$$\int_0^\varepsilon\frac{d\theta\, e^{-\theta}}{1-\hat{\mu}(\chi\theta)+k\sqrt{\sigma}\hat{\mu}(\chi\theta)}\approx\int_0^\varepsilon\frac{d\theta\, e^{-\theta}}{-\hat{\mu}'(\chi\theta)+C\sqrt{\sigma}} \approx\frac{\ln\sqrt{\sigma}}{\hat{\mu}'(0)\chi} \ ,$$

so that

$$\hat{\Lambda}(\sigma)\int_0^\infty e^{-\theta}\frac{d\theta}{1-\hat{\lambda}(\sigma)\hat{\mu}(\sigma+\chi\theta)}\approx\frac{K}{\hat{\mu}'(0)\chi}\frac{\ln\sqrt{\sigma}}{\sqrt{\sigma}} \ . \tag{A3}$$

2. Estimation of the second sum in (A1)

We use the bounds (A2). Let us consider the upper bound for small σ:

$$\hat{\lambda}(\sigma)\int_0^\infty e^{-\theta}\frac{\hat{M}(\sigma+\chi\theta)}{1-\hat{\lambda}(\sigma)\hat{\mu}(\sigma+\chi\theta)}d\theta \approx\int_0^\infty e^{-\theta}\frac{\hat{M}(\chi\theta)}{1-\hat{\mu}(\chi\theta)+K\sqrt{\sigma}\hat{\mu}(\chi\theta)} \ . \tag{A4}$$

The same computation as before shows that this is equivalent to $\ln\sigma$. Finally, using (A1), (A3), and (A4), we obtain

$$\hat{P}(\sigma)\approx\frac{K}{2\hat{\mu}'(0)\chi}\frac{\ln\sigma}{\sqrt{\sigma}} \ ,$$

from which we conclude that

$$P(L)\approx\frac{\ln L}{\sqrt{L}} \ .$$

APPENDIX B: ESTIMATION OF THE VELOCITY CORRELATION FUNCTION IN A STRONG FIELD

We take the Laplace transform of Eq. (18) taking into account the definition (20), Eq. (19), and (21):

$$\int_0^\infty e^{-\sigma t}\langle v(t)\rangle dt=\sum_{n\geq 0}\hat{\lambda}(\sigma)[\hat{\tau}(\sigma)\hat{\lambda}(\sigma)]^n + \sum_{n\geq 0}\hat{v}(\sigma)\hat{\lambda}(\sigma)[\hat{\tau}(\sigma)\hat{\lambda}(\sigma)]^n ,$$

from which we deduce Eq. (22). The problem is now to study the small σ behavior of the second member of Eq. (22). First of all, we know that

$$\hat{\lambda}(\sigma)=e^{-K\sqrt{\sigma}}\sim 1-K\sigma^{1/2}+K^2/2\sigma+O(\sigma^{3/2}) ,$$
$$\hat{\lambda}(\sigma)\frac{1}{\sigma}[1-\hat{\lambda}(\sigma)]\sim K\sigma^{-1/2}-K^2/2+O(\sigma^{1/2}) . \quad \text{(B1)}$$

Now, let us recall that, given L_{2k}, the exit time T_{2k} of the finite interval $[0,L_{2k}]$ is finite and its expectation is of order L_{2k}^2. As a consequence, using the definition (17), we have for small σ,

$$\hat{\tau}(\sigma)=1-\langle T_{2k}\rangle\sigma+O(\sigma) \quad \text{(B2)}$$

and

$$\langle T_{2k}\rangle=\int_0^\infty \mu(l)\langle T_{2k}|L_{2k}=l\rangle dl . \quad \text{(B3)}$$

But $\mu(l)$ is the distribution of the exit time of a Brownian particle from the finite interval $[-R_0,+R_0]$ and has moments of all orders, so that (B2) is finite. Moreover, we have from Eq. (16),

$$\hat{v}(0)=\left\langle\int_0^\infty v(t)\eta(\{t\leq T_{2k}\})dt\right\rangle =\left\langle\int_0^{T_{2k}} v(t)dt\right\rangle=\langle L_{2k}\rangle=\int_0^\infty l\mu(l)dl ,$$

which is also finite. From (Eqs. (22), (B1), (B2), and (B4), we can easily obtain the small σ asymptotic expansion

$$\left\langle\int_0^\infty e^{-\sigma t}v(t)dt\right\rangle=\frac{1}{\sigma}+\frac{\langle L_{2k}\rangle-\langle T_{2k}\rangle}{K}\frac{1}{\sqrt{\sigma}}+\cdots ,$$

from which we deduce Eq. (23).

APPENDIX C: ESTIMATION OF THE CONSTANT IN EQ. (23)

We want to estimate the constant in Eq. (23). Let us recall that q_n performs a standard random walk on a lattice with spacing $\sim\beta^{-1/2}$ and R_0 is some distance in this lattice: $R_0=N\beta^{-1/2}$ for integer N. Then $\lambda(l)$ is the density probability of the first time that, starting from $R_0+\beta^{-1/2}$, the walker q_n comes back to R_0. This means that $\hat{\lambda}(\sigma)\sim\exp(-K\sigma^{1/2})$ and K is an absolute constant. Now, the conditional expectation $\langle T_{2k}|L_{2k}=l\rangle$ is the exit time from the interval $[0,l]$ of a diffusing particle starting from 0 and reflected back at 0 each time it comes back to 0, so that it is $l^2/2$.

Then,

$$\langle T_{2k}\rangle=\int_0^\infty \frac{l^2}{2}\mu(l)dl , \quad \text{(C1)}$$

where $\mu(l)dl$ is the probability distribution of L_{2k}. Thus, $\mu(l)dl$ is the probability that a random walk with unit step, starting from $N-1$, leaves the interval $[N,+N]$ for the first time in $[l,l+dl]$. More generally, let us call $\mu_x(l)dl$ the probability that, starting from $-N\leq x\leq +N$, the first exit time from $[-N,+N]$ is in $[l,l+dl]$ and define

$$\hat{\mu}_x(\sigma)=\int_0^\infty e^{-\sigma l}\mu_x(l)dl .$$

Then, $\hat{\mu}_x(\sigma)$ satisfies

$$\frac{\partial^2\hat{\mu}_x(\sigma)}{\partial u^2}=2\sigma\hat{\mu}_x(\sigma) ,$$
$$\hat{\mu}_x(\sigma)=1 \quad \text{for } x=-N \text{ or } +N ,$$

so that

$$\hat{\mu}_x(\sigma)=\frac{\cosh x\sqrt{2\sigma}}{\cosh N\sqrt{2\sigma}} \quad \text{(C2)}$$

and

$$\hat{\mu}(\sigma)=\frac{\cosh(N-1)\sqrt{2\sigma}}{\cosh N\sqrt{2\sigma}} . \quad \text{(C3)}$$

From Eq. (C1), $\langle T_{2k}\rangle$ is proportional to the second derivative of $\hat{\mu}(\sigma)$ at $\sigma=0$ and this is easily seen to be $\frac{2}{3}N^3$. Moreover, $\langle L_{2k}\rangle=\int_0^\infty l\mu(l)dl$ and is only of order N^2. This proves that for large R_0,

$$\langle v(t)\rangle-1\sim-\frac{C(R_0\beta^{1/2})^3}{\sqrt{t}} ,$$

where C is an absolute positive constant.

[1] G. A. van Velzen, thesis, Utrecht University, 1990; M. H. Ernst and G. A. van Velzen, J. Stat. Phys. **57**, 455 (1989).
[2] P. M. Binder, Complex Systems **1**, 559 (1987).
[3] H. van Beijeren and M. H. Ernst, J. Stat. Phys. **70**, 793 (1993).
[4] H. van Beigjeren, Rev. Mod. Phys. **54**, 195 (1982).
[5] Th. M. Nieuwenhuizen and M. H. Ernst, Phys. Rev. B **31**, 3518 (1985); J. van Polen and H. van Beijeren, Physica A **170**, 247 (1991).
[6] M. Frankowicz, B. Gaveau, and M. Moreau, Phys. Lett. A **152**, 262 (1991).
[7] A. Perera, B. Gaveau, M. Moreau, and K. Penson, Phys. Lett. A **159**, 152 (1991).
[8] M. Moreau, B. Gaveau, M. Frankowicz, and A. Perera, in *Far-From-Equilibrium Dynamics of Chemical Systems*, edited by J. Popielawski and J. Gorecki (World Scientific, Singapore, 1991).
[9] X. P. Kong and E. G. D. Cohen, Phys. Rev. B **40**, 4838

(1989); Physica D **47**, 9 (1991).
[10] J. P. Hansen and I. MacDonald, *Theory of Simple Liquids* (Academic, London, 1976).
[11] H. P. MacKean, *Stochastic Integrals* (Academic, London, 1968).
[12] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987); J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 129 (1991); S. F. Burlatsky and G. Oshanin, J. Stat. Phys. **65**, 1095 (1991); **65**, 1109 (1991).